An Efficient Method for Analysis of Arbitrary Nonuniform Transmission Lines

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***Abstract*— The analytical solution of an ideal linear varied nonuniform transmission line (LNTL) has been obtained and the exact linear two-port ABCD matrix of LNTL has been given correctly for the first time. By using cascaded LNTL sections to approximate an arbitrary characteristic impedance profile, a new technique has been presented in this paper for analyzing an arbitrary nonuniform transmission line (NTL). The technique is far better than the conventional technique in terms of the computational accuracy and intensity since it uses a piecewise-linear characteristic impedance profile in place of the stepped profile used by the conventional technique. Several numerical examples have been given to demonstrate the method.**

I. Introduction

Nonuniform transmission lines (NTL's) have been widely used by microwave engineers in many applications, such as impedance matching [1], pulse shaping [2], and analog signal processing [3]. NTL's also exist in many VLSI interconnection structures to provide smooth connections between high-density IC chips and their chip carriers [4], [5]. All of these applications requires an efficient method for analyzing NTL's. Many techniques have been developed for analyzing NTL's in both the frequency and time domains over the past 50 years. The reflection coefficient or the voltage/current along a NTL can be described by a differential equation. Unfortunately, this differential equation is a nonlinear Riccati-type equation and its general solution does not exist analytically [1]. For several special types of ideal NTL such as exponential NTL's (ENTL) [1] and power-law NTL's (PNTL) [6], the corresponding nonlinear Riccati equations can be solved analytically without approximation. For the general case, approximations such as the assumption that the nonlinear part of the Riccati equation is negligible are normally needed to obtain the solutions [1], [7], [11], [14]. Due to the lack of general analytical solutions, numerical techniques are commonly used for analysing NTL's [4], [5], [9], [10]. Most of these techniques treat a NTL as a cascaded combination of many small uniform transmission line (UTL) sections [see Fig. 1(a)], [4], [5], [10]. The accuracy and the computational intensity of these techniques increase sharply as the number of small sections increase. The inefficiency of these techniques is clear since the stepped-type of impedance profile is a poor approximation to the real continuously varied profile of a general NTL.

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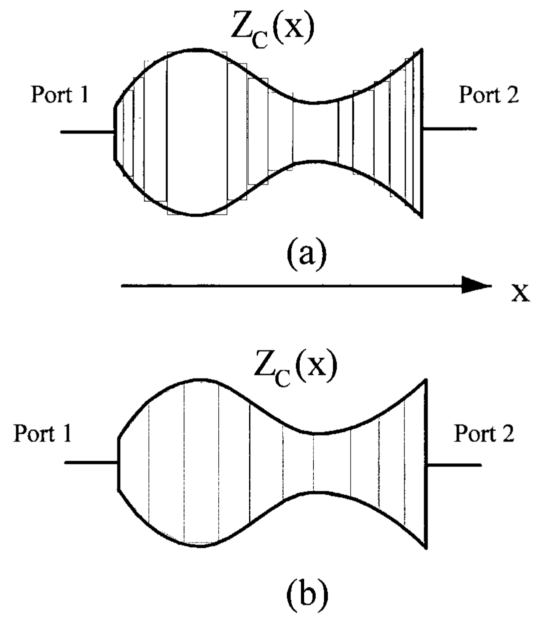


Fig. 1. Approximate an arbitrary nonuniform transmission line by using (a) a group of cascaded UTL sections and (b) a group of cascaded LNTL sections.

In this paper, the analytical solution of an ideal linear varied NTL (LNTL) has been derived and the *exact* linear two-port ABCD matrix of the ideal LNTL has been presented correctly for the first time. Based on this ABCD matrix, a new technique for analysing general NTL is proposed to replace the conventional techniques [4], [5], [10]. The novelty of the new technique is that it uses cascaded ideal LNTL sections to approximate the impedance profile of an arbitrary NTL [Fig. 1(b)]. Since the discontinuous impedance profile in Fig. 1(a) is replaced by a smoother impedance profile in Fig. 1(b), the new technique is far better than the conventional approaches in terms of computational efficiency and accuracy.

II. Analytical Solution for an LNTL

Consider an ideal lossless transmission line (Fig. 2). Its characteristic impedance and propagation coefficient are

$$Z_c(x) = Z_c(0)(1 + k \cdot x), \qquad 0 \le x \le L \tag{1a}$$

$$\beta_0 = \frac{\omega}{c} \tag{1b}$$

where x is the position along the line, k is the slope constant, L is the length of the line, ω is angular frequency, and c is the velocity of the light. The differential equation (generalized

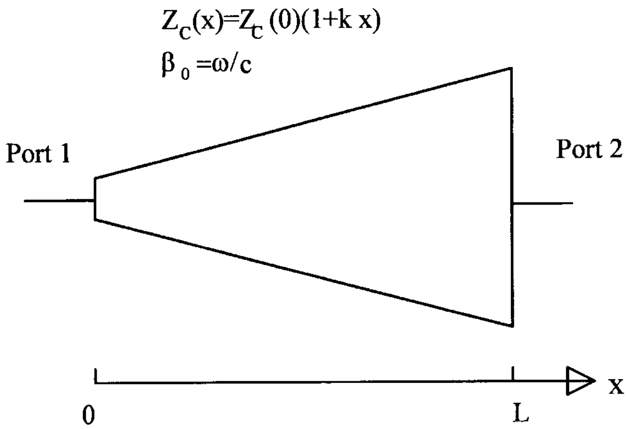


Fig. 2. An LNTL section.

telegraph equation) describing this system can be written as

$$\frac{d^2V(x)}{dx^2} - \frac{k}{1+kx}\cdot\frac{dV(x)}{dx} + \beta_0^2\cdot V(x) = 0 \qquad (2a)$$

$$\frac{d^2I(x)}{dx^2} + \frac{k}{1+kx}\cdot\frac{dI(x)}{dx} + \beta_0^2\cdot I(x) = 0 \qquad (2b)$$

which is an analytically solvable equation [8], [12], [13]. Although the solvability of (2) is known [8], [12], [13], the way to obtain the linear two-port parameters of an ideal LNTL is not easy due to the complex algebraic operations with the involvement of Bessel functions. Rustogi [13] has obtained the ABCD parameters of this transmission line system by solving (2). Unfortunately, the ABCD parameters given in [13] have several obvious errors. The author of [13] also has not provided details concerning the solution procedure. In the present work, (2) has been solved again independently and the details have been given in the Appendix. Using this result, the ABCD matrix for the above ideal LNTL can be written as

$$A = \frac{\pi\beta_0}{2k}[J_1(u_1)Y_0(u_2) - J_0(u_2)Y_1(u_1)] \qquad (3a)$$

$$B = -j\frac{\pi\beta_0}{2k}Z_c(0)(1+kL)[J_1(u_2)Y_1(u_1) - J_1(u_1)Y_1(u_2)] \qquad (3b)$$

$$C = j\frac{\pi\beta_0}{2kZ_c(0)}[J_0(u_1)Y_0(u_2) - J_0(u_2)Y_0(u_1)] \qquad (3c)$$

$$D = -\frac{\pi\beta_0}{2k}(1+kL)[J_0(u_1)Y_1(u_2) = J_1(u_2)Y_0(u_1)] \qquad (3d)$$

where $u_1 = \beta_0/k, u_2 = (1+kL)u_1, J_n(x)$ is the Bessel function of the first kind of order n and $Y_n(x)$ is the Bessel function of the second kind of order n. Other matrix parameters such as S, Y, and Z can also be obtained by using matrix conversion. It is worth noting that (3) is the *exact* solution of the ideal LNTL.

In the extreme case where the slope constant of the ideal LNTL approaches zero corresponding to u_1 and u_2 approaching infinite, the LNTL should be identical to a uniform transmission line section. To prove that, one can use the following asymptotic forms of the Bessel functions to replace the Bessel functions in (3) (also see [12])

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\right) \qquad x \to +\infty \qquad (4a)$$

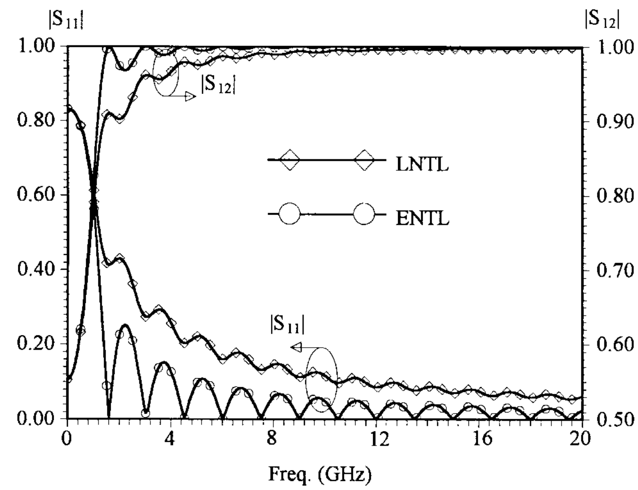


Fig. 3. Comparison of $|S_{11}|, |S_{12}|$ for an ENTL and a LNTL with same impedance ratio and length of the line ($Z_{\rm in} = Z_s = 25\ \Omega, Z_{\rm out} = Z_L = 275\ \Omega$, and $L = 0.1$ m).

$$Y_0(x) \sim \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\pi}{4}\right) \qquad x \to +\infty \qquad (4b)$$

$$J_1(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{3\pi}{4}\right) \qquad x \to +\infty \qquad (4c)$$

$$Y_1(x) \sim \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{3\pi}{4}\right) \qquad x \to +\infty. \qquad (4d)$$

In doing this, (3a)–(d) reduce to

$$A = \cos(\beta_0 L) \qquad k \to 0 \qquad (5a)$$

$$B = jZ_c(0)\sin(\beta_0 L) \qquad k \to 0 \qquad (5b)$$

$$C = j\frac{\sin(\beta_0 L)}{Z_c(0)} \qquad k \to 0 \qquad (5c)$$

$$D = \cos(\beta_0 L) \qquad k \to 0 \qquad (5d)$$

which are identical to the ABCD parameters of a uniform transmission line section, as we expected.

Fig. 3 shows reflection coefficients for an ideal LNTL section and an ideal ENTL section, both having the same length and output/input impedance ratio. Fig. 3 also clearly demonstrates that the LNTL exhibits the high-pass behavior. The LNTL also has a higher value of reflection coefficient, but less ripple over the whole frequency band in comparison to the ENTL.

III. Proposed Technique for Analysis of Arbitrary NTL's T

For an arbitrary NTL (such as a nonuniform stripline) (Fig. 1), both its characteristic impedance and propagation coefficient are position-dependent and can be expressed as

$$Z_c(x) = Z_c(0)\cdot f(x), \qquad 0 \le x \le L \qquad (6)$$

$$\beta(x,\omega) = \frac{\omega}{c}\cdot g(x) \qquad ,0 \le x \le L \qquad (7)$$

where $f(x)$ and $g(x)$ are arbitrary functions that have positive values and are spatially continuous along the NTL. The complex reflection coefficient $\Gamma(x)$ of the NTL is determined by the differential equation [1]:

$$\frac{d\Gamma(x)}{dx} - 2 \cdot j\beta(x) \cdot \Gamma(x) + (1 - \Gamma^2(x)) \cdot K(x) = 0 \quad \text{(8a)}$$

$$K(x) = \frac{1}{Z_c(x)} \cdot \frac{dZ_c(x)}{dx} \quad \text{(8b)}$$

$$\Gamma(x) = \frac{V(x) - Z_c(x) \cdot I(x)}{V(x) + Z_c(x) \cdot I(x)}. \quad \text{(8c)}$$

Equation (8a) is a Riccati equation that is nonlinear through the term $\Gamma^2(x)$. It is also worth indicating that by using voltage $V(x)$ and current $I(x)$ along the NTL in place of the $\Gamma(x)$ in (8) one can also obtain another set of equations (generalized telegraph equations) which are similar to (2).

By introducing a new variable

$$y(x) = \int_0^x g(z)\, dz \qquad 0 \le x \le L \quad \text{(9)}$$

(8) can be further simplified to

$$\frac{d\Gamma(y)}{dy} - 2 \cdot j\beta_0 \cdot \Gamma(y) + (1 - \Gamma^2(y)) \cdot K(y) = 0 \quad \text{(10a)}$$

$$K(y) = \frac{1}{Z_c(y)} \cdot \frac{dZ_c(y)}{dy} \quad \text{(10b)}$$

where $\beta_0 = \omega/c$. From (10), it is clear that $\Gamma(y)$ is uniquely determined by the characteristic impedance profile $Z_c(y)$ and the boundary conditions at ports 1 and 2. In (10), the propagation constant is no longer position-dependent under the new variable y. For an arbitrary $Z_c(y)$, the piecewise-linear approximation can always be applied. Based on the above analysis, the proposed technique is carried out by the following procedure.

Step 1) Numerically evaluate $y(x)$ by using (9).

Step 2) Select a set of points $(x_0, x_1, \cdots, x_i, \cdots, x_n)$ (corresponding to a set of points in terms of $y(x), (y_0, y_1, y_2, \cdots, y_i, \cdots, y_n)$ in the region $[0, L]$) so the NTL is broken to n small sections. The $Z_c(y)$ of the ith small section can be approximated by an LNTL

$$Z_c(y) \approx \hat{Z}_c(y) = Z_c(y_i)(1 + k_i(y - y_i))$$
$$y_i \le y \le y_{i+1} \quad \text{(11)}$$
$$k_i = \frac{Z_c(y_{i+1}) - Z_c(y_i)}{Z_c(y_i) \cdot (y_{i+1} - y_i)} \quad \text{(12)}$$

where $i = 0, 1, 2, \cdots, n-1$.

Step 3) Evaluate the ABCD matrix $[A^{i\text{th}}]$ of the ith section by using $L_i = y_{i+1} - y_i, k_i, Z_c(y_i)$ in place of L, k, and $Z_c(0)$ in (3).

Step 4) Evaluate the ABCD matrix $[A^{\text{Total}}]$ by cascading the $[A^{i\text{th}}]$ for all small sections.

$$[A^{\text{Total}}] = \prod_{i=1}^{n} [A^{i\text{th}}]. \quad \text{(13)}$$

After $[A^{\text{Total}}]$ is obtained, the S, Z, or Y parameters can be calculated by using matrix conversion.

For a transmission line with dispersion, the $f(x)$ and $g(x)$ in (6) and (7) are also dependent on frequency

$$Z_c(x, \omega) = Z_c(0) \cdot f(x, \omega), \qquad 0 \le x \le L \quad \text{(14)}$$
$$\beta(x, \omega) = \frac{\omega}{c} \cdot g(x, \omega), \qquad 0 \le x \le L. \quad \text{(15)}$$

To analyze such a transmission line, the above procedure needs to be repeatedly performed at every single frequency point to get the broadband $[A^{\text{Total}}]$. The above method can also be extended to the lossy NTL, which will be the subject of another paper [16].

IV. Simulation Examples

The first example is an ideal exponential NTL (ENTL), which has an exact analytical solution (since this is only a numerical example, no attention has been paid to how one would physically fabricate this line). Fig. 4 shows the simulation results for the S_{11} and S_{12} calculated in four different ways: 1) from the exact solution; 2) using a ten-LNTL approximation; 3) using a ten-uniform transmission line section (UTL) approximation; and 4) using a 20-UTL approximation. The broken point set $(x_0, x_1, \cdots, x_i, \cdots, x_n)$ is chosen in a similar way for all these cases in order to satisfy

$$Z_c(x_{i+1}) - Z_c(x_i) = Z_c(x_i) - Z_c(x_{i-1}) \quad \text{(16)}$$

where $i = 1, 2, \cdots n$. For the ith UTL, the line parameters are

$$\overline{Z}_c^{i\text{th}} = \frac{Z_c(x_i) - Z_c(x_{i-1})}{2} \quad \text{(17)}$$
$$L_i = x_i - x_{i=1} \quad \text{(18)}$$

where $i = 1, 2, \cdots, n$.

Excellent agreement (up to 50 GHz or even higher) between the results from the analytical solution and the results from a ten-LNTL approximation is obtained [Fig. 4(a) and (b)]. In contrast with a 10-LNTL approximation, a 10-UTL approximation can only produce acceptable agreement with the analytical solution below about 3 GHz [Fig. 4(c) and (d)]. By using a 20-UTL approximation, the region of good agreement is only expanded to 6 GHz [Fig. 4(c) and (d], the poor agreement above 6 GHz makes the presentation above 10 GHz meaningless).

The second example is a microstrip linear taper. The width of the line is varied linearly from 0.1 to 5.1 mm, the dielectric constant of the substrate is 10.1, and the height of the substrate and the length of the line is 1.0 and 30 mm, respectively. To simplify the problem, the line is considered to be both lossless and dispersionless. The effective dielectric constant $\varepsilon_{\text{eff}}(x)$ characteristic impedance $Z_c(x)$, and $y(x)$ using the formulas given in [15] are shown in Fig 5. Fig. 5 shows that the impedance profile of a microstrip linear taper is not a linear function at all. The simulation results are shown in Fig. 6. Three different ways are used to approximate the microstrip linear taper: 1) a ten-LNTL approximation; 2) a ten-UTL approximation; and 3) a 20-UTL approximation. It is clear from Fig. 6 that the agreement between a 20-UTL and a 10-LNTL is better than the agreement between a 10-UTL and a 10-LNTL for both S_{11} and S_{12}. Since the structure should

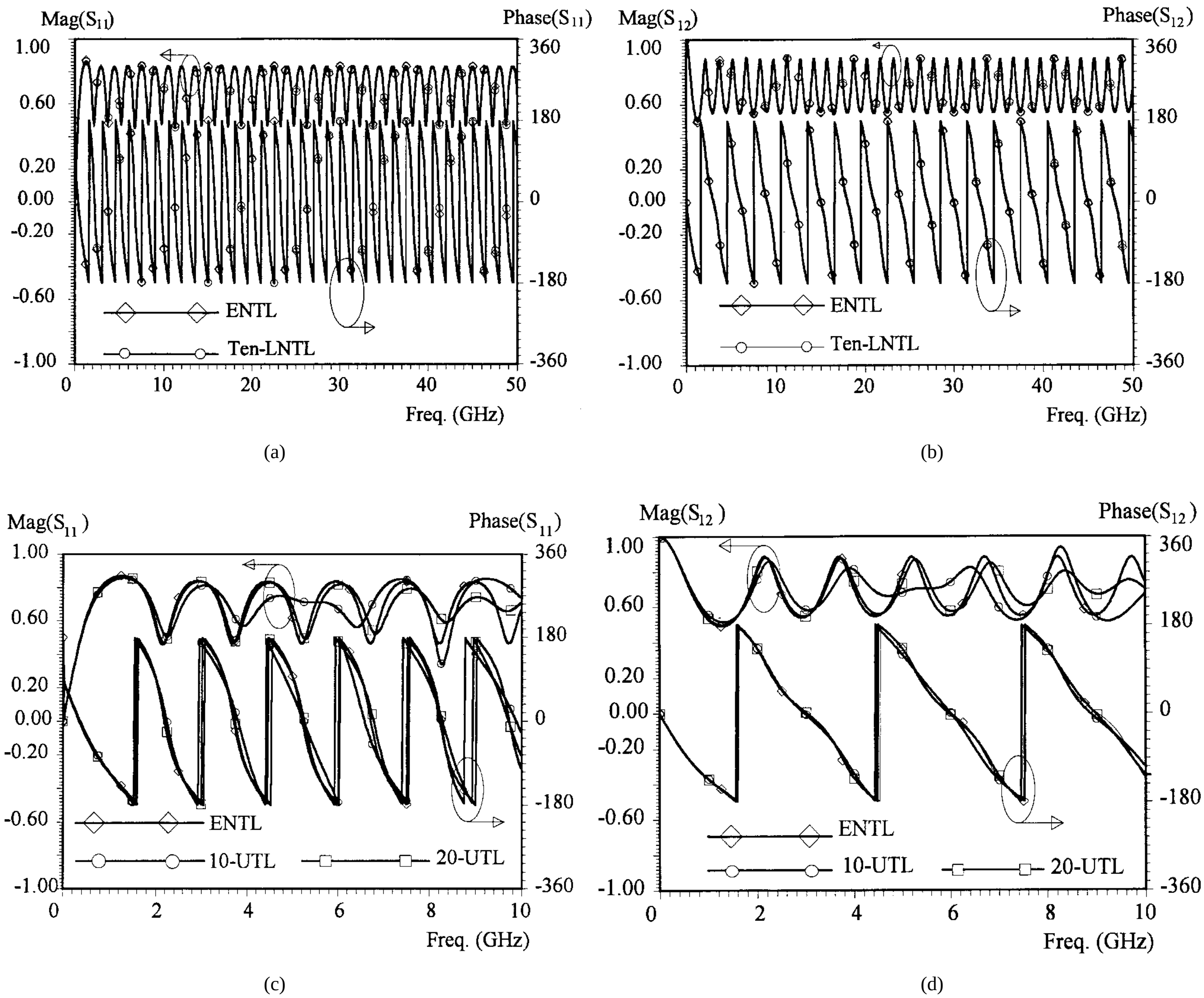


Fig. 4. Simulated $|S_{11}|$, phase $(S_{11}), |S_{12}|$, and phase (S_{12}) of an ideal ENTL ($Z_{\rm in} = 25\ \Omega, Z_{\rm out} = 275\ \Omega$, and $L = 0.1$ m, $Z_s = Z_L = 50\ \Omega$). (a) Comparison of $|S_{11}|$ and phase (S_{11}) simulation results by using ENTL analytical solution and a 10-LNTL approximation. (b) Comparison of $|S_{12}|$ and phase (S_{12}) simulation results by using ENTL analytical solution and a ten-LNTL approximation. (c) Comparison of $|S_{11}|$ and phase (S_{11}) simulation results by using ENTL analytical solution, a 10-UTL approximation, and a 20-UTL approximation. (d) Comparison of $|S_{12}|$ and phase (S_{12}) simulation results by using ENTL analytical solution, a 10-UTL approximation, and a 20-UTL approximation.

present a high-pass behavior according to theory, the ten-LNTL is the best approximation. Finally, it is worth noting that at least 70–250 small UTL sections are needed to approximate a practical microstrip taper [10].

V. Conclusion

This paper presents an efficient technique for analyzing an arbitrary NTL by piecewise-linearly approximating the characteristic impedance profile of the line. The ABCD parameters of the ideal LNTL section are obtained here without approximation. By using these parameters, the whole NTL ABCD parameters are evaluated by using matrix multiplication. The new technique is better than the stepped approximation (which has been widely accepted by microwave engineers) due to the associated computational efficiency and accuracy. The advantage of the technique is obvious since the piecewise-linear approximation is intrinsically better than the stepped approximation. The new technique can be easily implemented in the CAD environment.

Appendix

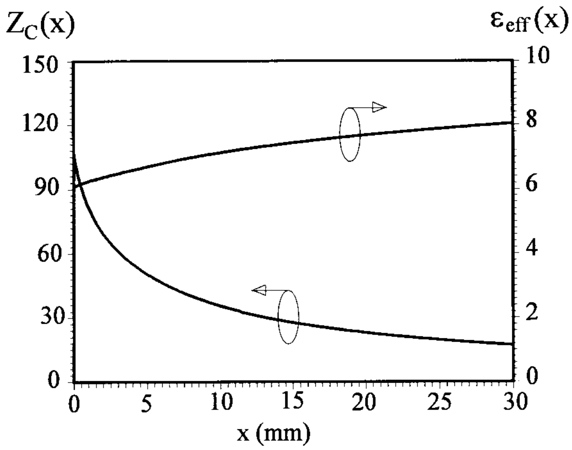
Equation (2a) can be solved according to the following procedure.

Step 1) By introducing a new variable $y = 1 + kx$, (2a) can be rewritten as

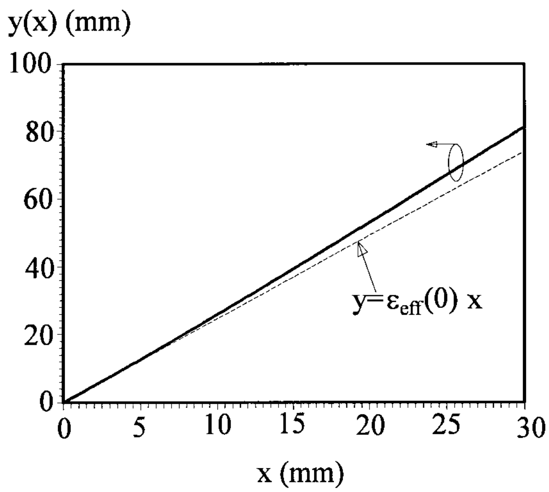
$$\frac{d^2V(y)}{dy^2} - \frac{1}{y}\cdot\frac{dV(y)}{dy} + \frac{\beta_0^2}{k^2}\cdot V(y) = 0. \qquad \text{(A.1)}$$

Step 2) Let $V(y) = y^n \cdot U(y)$ and substitute it into (A.1); then

$$y^n \cdot \frac{d^2U(y)}{dy^2} + (2n-1)\cdot y^{n-1}\cdot\frac{dU(y)}{dy} + \left[n(n-1) - n + \frac{\beta_0^2}{k^2}\right]\cdot y^{n-2}\cdot U(y) = 0. \quad \text{(A.2)}$$



(a)



(b)

Fig. 5. Calculated microstrip parameters as the functions of the position. (a) $\varepsilon_{\rm eff}(x)$ and $Z_c(x)$. (b) $y(x)$.

Step 3) For $n = 1$ and letting $z = (\beta_0/k)y$, (A.2) is further simplified to

$$\frac{d^2U(z)}{dz^2} + \frac{1}{z} \cdot \frac{dU(z)}{dz} + \left(1 - \frac{1}{z^2}\right) = 0 \qquad \text{(A.3)}$$

which is a first-order Bessel equation and has a general solution:

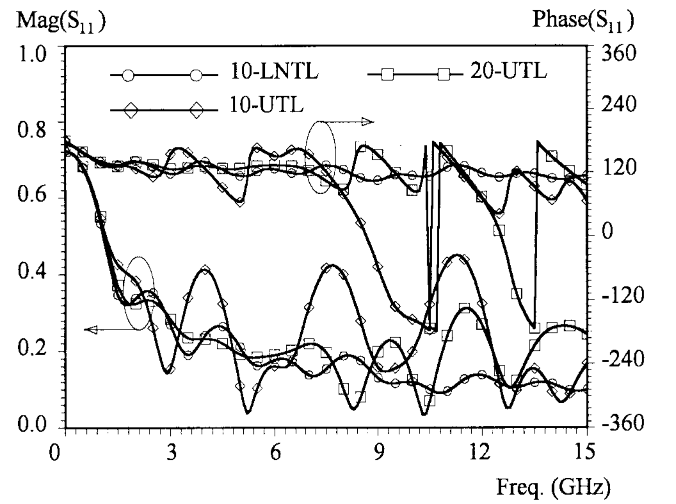
$$U(z) = K_1 \cdot J_1(z) + K_2 \cdot Y_1(z) \qquad \text{(A.4)}$$

where K_1 and K_2 are the constants, which can be determined by the boundary conditions, and $J_n(z)$ and $Y_n(z)$ are the Bessel functions of the first kind of order n and the second kind of order n, respectively.

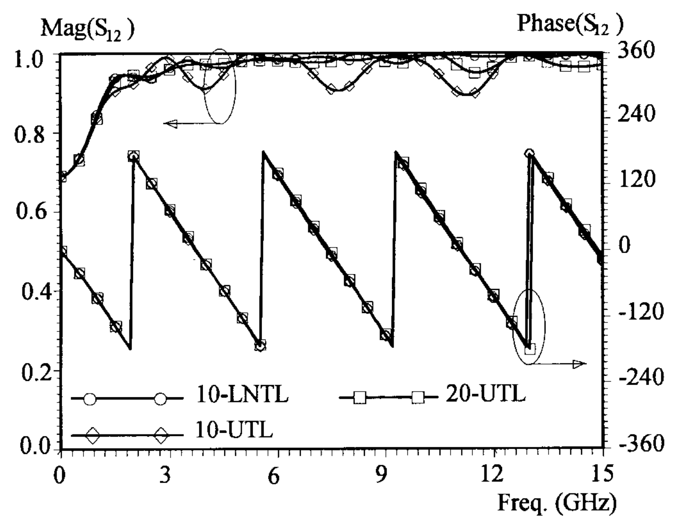
Step 4) So, the final solution for $V(x)$ is

$$V(x) = (1+kx)\left\{K_1 \cdot J_1\left[\frac{\beta_0}{k}(1+kx)\right] + K_2 \cdot Y_1\left[\frac{\beta_0}{k}(1+kx)\right]\right\}. \qquad \text{(A.5)}$$

Since $I(x) = -[1/Z_c(x)]\,[dV(x)/dx]$, the solution for $I(x)$ can also be obtained by using (A.5).



(a)



(b)

Fig. 6. Simulated $|S_{11}|$, phase $(S_{11}), |S_{12}|$, and phase (S_{12}) of an microstrip taper. (a) Comparison of $|S_{11}|$ and phase (S_{11}) simulation results by using a ten-LNTL approximation, a 10-UTL approximation, and a 20-UTL approximation. (b) Comparison of $|S_{12}|$ and phase (S_{12}) simulation results by using a 10-LNTL approximation, 10-UTL approximation, and a 20-UTL approximation.

Step 5) The ABCD matrix for a piece of LNTL is obtained by applying proper boundary conditions to $V(x)$ and $I(x)$. The final results are given in (3).

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